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Exact Analytic Results for Composite Fermions in a Rajaraman-Sondhi like formulation

Sumathi Rao [1]
Mehta Research Institute, Chhatnag Road,Jhunsi,
Allahabad 211 019, India

Abstract

We obtain the exact spectrum and the unique ground state of two composite fermions (in a Rajaraman - Sondhi like formulation) in an external magnetic field B. We show that the energy eigenvalues decrease with increasing angular momentum, thus making it energetically favourable for composite fermions to stay apart. Generalising this result to a gas of composite fermions, we provide an energetic justification of the Laughlin and Jain wavefunctions.

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[1] *e-mail address*: sumathi@mri.ernet.in

Experimental discoveries of new heterojunctions[1] is one reason for the continued interest in the phenomenon of the fractional quantum Hall effect (FQHE)[2]. The other important reason is that theoretically, the FQHE has been unusually fruitful in giving rise to novel ideas and excitations, such as composite fermions[3], skyrmions[4], a new kind of non-Fermi liquid at $\nu = 1/2$[5], etc.

Composite fermions are now well-established[6] as the relevant quasi-particles in the fractional quantum Hall effect (FQHE) system. True to the nature of quasi-particles, they are weakly interacting, and in fact, the main features of the FQHE phenomenon can be simply understood in terms of a model of non-interacting composite fermions - *i.e.*, FQHE is merely the integer QHE of non-interacting composite fermions. Composite fermions were originally introduced in the microscopic, first quantised trial wave-function approach[3] by Jain. He attached Jastrow factors to IQHE wave-functions to get FQHE wave-functions. By interpreting the Jastrow factors as even units of flux quanta attached to the electrons, he showed that FQHE of electrons at fractions $n/(2mn+1)$ is equivalent to IQHE of composite electrons (electrons with $2m$ flux units attached) at level n. On a different front, Zhang et al[7] formulated a field theory of the FQHE in terms of a Chern-Simons (CS) gauge field, where the electrons were interpreted as bosons with odd number of flux quanta. Later, the original composite fermion idea of mapping a system of strongly interacting fermions in a magnetic field, to a system of weakly interacting composite fermions was itself implemented as a CS field theory[8]. The composite fermions in the CS model included the phase due to the flux quanta attached to the fermions, but not its amplitude. A more recent approach by Rajaraman and Sondhi[9] remedies this defect, but at the expense of making the gauge field complex.

In this letter, we study a system of two composite fermions in an external magnetic field. We use a Rajaraman-Sondhi like formulation to model the composite fermions - *i.e.*, our composite fermions are ordinary fermions interacting with the complex vector potential introduced by Rajaraman and Sondhi in Ref.[9]. Thus, the quantum mechanical problem reduces to that of two fermions interacting with a complex vector field and an external magnetic field. We obtain the energy eigenvalues and the wave-functions and contrast them with the spectrum obtained by Chern-Simons (CS) composite fermions (composite fermions modelled by interaction with a Chern-Simons gauge field). The CS composite fermions behave just like usual fermions.

They have a large angular momentum degeneracy in the presence of an external magnetic field and no unique ground state. However, for our two composite fermions, this degeneracy breaks. The energy decreases as a function of the angular momentum, so the minimum energy solution is obtained for the maximum value of angular momentum. For the two composite fermion system, the maximum is set by the ratio of the size of the system to the magnetic length. Hence, the maximum of the angular momentum increases with decrease in magnetic length or equivalently increase in the external magnetic field. We argue, hence, that the wave-function for many composite fermions should be an eigenstate of maximum possible angular momentum.

The Hamiltonian for two composite fermions in an external magnetic field is given by

$$H = \sum_i^2 \frac{(\mathbf{p}_i - e\mathbf{v}_i - e\mathbf{A}_i)^\dagger (\mathbf{p}_i - e\mathbf{v}_i - e\mathbf{A}_i)}{2m} \tag{1}$$

where $\mathbf{A}_i = B/2(-y_i, x_i)$ are the vector potentials of the external magnetic field B in the $\hat{z}$ direction and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the Rajaraman-Sondhi (RS) complex gauge fields seen by each of the composite fermions due to the presence of the vortex in the other composite fermion. As explained in Ref.[9], the complex gauge field $\mathbf{v}_i$ is related to the usual CS gauge field $\mathbf{a}_i$ as

$$\mathbf{v}_i = \mathbf{a}_i + i\hat{z} \times \mathbf{a}_i. \tag{2}$$

As is well-known from anyon studies (see for example Ref.[10]), for a two particle system, the CS gauge field is given by

$$\mathbf{a}_1 = \frac{\alpha}{\pi e} \frac{\hat{z} \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}, \quad \mathbf{a}_2 = \frac{\alpha}{\pi e} \frac{\hat{z} \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_1 - \mathbf{r}_2|^2}. \tag{3}$$

Here α is the statistics parameter, which for composite fermions is an even integer $\times\pi$. The complex gauge fields $\mathbf{v}_1$ and $\mathbf{v}_2$ are completely defined in terms of the CS field. The imaginary term in $\mathbf{v}_i$ is the radial component in the gauge field and represents a 'fat flux' or spread-out flux centred at the position of the fermion. (The pure CS gauge field, in contrast, attaches infinitesimal flux-tubes to the fermion thereby only changing its phase.)

Note that this Hamiltonian is not the same as the Hamiltonian considered in Ref.[9]. Their field theoretic Hamiltonian was given by

$$H_{RS} = \int d^2x \Pi(\mathbf{x}) \frac{(\mathbf{p} - e\mathbf{v} - e\mathbf{A})^2}{2m} \chi(\mathbf{x}) \tag{4}$$

where $\chi(\mathbf{x})$ denoted the composite fermion field and $\Pi(\mathbf{x})$ was the canonically conjugate Fermi field. However, $\Pi(\mathbf{x}) = \chi^\dagger(\mathbf{x})e^{(J(\mathbf{x})+J^\dagger(\mathbf{x}))} \neq \chi^\dagger(\mathbf{x})$, where $J(\mathbf{x})$ is related to the complex vector potential as $\mathbf{v} = i\vec{\nabla}J/e$. Thus, although $\mathbf{v}$ is complex, the Hamiltonian in Eq.(4) *is* hermitean. Note also that the RS gauge field in Ref.[9] also included a c-number term involving the Gaussian factor $exp(-r^2/4l^2)$, which was needed in their field theoretic formulation, since the field theory is defined for fixed area. Here, we drop the Gaussian factor in the gauge field and follow the more common practice of incorporating the whole Gaussian factor in the IQHE wave-function, with the Gaussian evaluated at the external magnetic field[11]. Nevertheless, the motivation for the Hamiltonian in Eq.(1) does come from the RS field theory and we incorporate its main feature, which is that the gauge field includes the amplitude as well as the phase of the composite fermions.

The CM motion which just represents a particle with mass $2m$ in twice the external magnetic field can be trivially factored out. We are then left with the one-particle relative Hamiltonian given by

$$H_{\rm rel} = \frac{(\mathbf{p} - e\mathbf{v} - e\mathbf{A}_{\rm rel})^\dagger(\mathbf{p} - e\mathbf{v} - e\mathbf{A}_{\rm rel})}{m} \tag{5}$$

with $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$, $\mathbf{v} \equiv \mathbf{v}_{\rm rel} = \mathbf{a}_{\rm rel} + i\hat{z} \times \mathbf{a}_{\rm rel}$ where $\mathbf{a}_{\rm rel}$ in turn is given by $\mathbf{a}_{\rm rel} = (\alpha/\pi e)(\hat{z} \times \mathbf{r}/|\mathbf{r}|^2)$ and $\mathbf{A}_{\rm rel} = B/4(-y, x)$. The wave-function is separable in cylindrical coordinates $\psi(r,\theta) = R(r)Y(\theta)$ and we find that the radial equation reduces to

$$[-(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}) + \frac{1}{r^2}[(L-\alpha/\pi)^2 + (\alpha/\pi)^2] - \frac{1}{l^2}(L-\alpha/\pi) + \frac{r^2}{4l^4} - mE]R(r) = 0 \tag{6}$$

where $l^2 = 2/eB$ is the magnetic length and L is the angular momentum - *i.e.*, the angular part of the wave-function is $Y(\theta) = e^{iL\theta}$. Fermi statistics of the composite fermions require that L be an odd integer. The solution of the above radial equation can be explicitly found in terms of a confluent hypergeometric function F,

$$R(r) = r^s e^{-r^2/4l^2} F(\frac{1}{2}[s - (L - \alpha/\pi) + 1] - k, s+1, \frac{r^2}{2l^2}) \tag{7}$$

with $s = \sqrt{(L-\alpha/\pi)^2 + (\alpha/\pi)^2}$, and $k = mEl^2/2$. The requirement that the series solution for the confluent hypergeometric function terminate leads

to the energy eigenvalues

$$E_{n,L} = \frac{eB}{2m}(\sqrt{(L-\alpha/\pi)^2 + (\alpha/\pi)^2} - (L-\alpha/\pi) + 1 + 2n) \tag{8}$$

where n is an integer. For the ground state, $n = 0$.

Before analysing this solution, let us obtain the equivalent solution when the RS gauge field is replaced by a CS gauge field, for contrast. The CS Hamiltonian is given by

$$H_{\rm rel}^{\rm CS} = H_{\rm rel} - \frac{\alpha^2}{\pi^2 r^2} \tag{9}$$

The subtracted term on the RHS is the one that appeared due to the radial terms in the RS gauge potential. Clearly, the net effect of the radial term has just been to increase the centrifugal barrier. For this Hamiltonian, the solution is simply

$$\begin{aligned} \psi^{\rm CS}(r,\theta) &= R^{\rm CS}(r)Y^{\rm CS}(\theta) && (10) \\ &= r^{|t|}e^{-r^2/4l^2}F(\frac{1}{2}[|t| - (t) + 1] - k, |t| + 1, \frac{r^2}{2l^2})e^{iL\theta} && (11) \end{aligned}$$

with $t = (L - \alpha/\pi)$ yielding the energy eigenvalues

$$E_{n,L}^{\rm CS} = \frac{eB}{2m}(|L-\alpha/\pi| - (L-\alpha/\pi) + 1 + n) \tag{12}$$

Here, however, by defining a new angular momentum $L' = L - \alpha/\pi$, we see that both the solution (except for an unobservable phase factor, since for composite fermions, α/π = even integer) and the energy eigenvalues reduce to that of an ordinary fermion in an external magnetic field. In particular, the massive degeneracy of a fermion in an external magnetic field (all positive values of L' are degenerate with $L' = 0$ [2]) persists for the CS composite fermion and there is no unique ground state. This is not surprising, since the CS composite fermion is just a gauge transform of the original fermion, albeit singular - *i.e.*, $\psi_{\rm CS}(\mathbf{r}_1 - \mathbf{r}_2) = (\mathbf{r}_1 - \mathbf{r}_2)\psi(\mathbf{r}_1 - \mathbf{r}_2)/|(\mathbf{r}_1 - \mathbf{r}_2)|$. (The same result would also be obtained if we study a naive first quantised picture of

[2] Note that a change in sign of the external magnetic field changes the sign of $L' = (L - \alpha/\pi)$. For the opposite sign of B, all negative values of L' are degenerate with $L' = 0$.

the RS field theory. The Hamiltonian is $H = \sum_i^2 (\mathbf{p}_i - e\mathbf{v}_i - e\mathbf{A}_i)^2/2m$ with the hermiticity of the Hamiltonian being maintained by defining a new inner product in the Hilbert space $<\psi|O|\phi> = \int \psi^* O \phi e^{-(J(\mathbf{x}) + J^\dagger(\mathbf{x}))} d^2x$, analogous to the field redefinitions made in Ref.[9]. The eigenvalues of this Hamiltonian also reduce to that of non-interacting fermions in an external magnetic field and the massive degeneracy remains unbroken. The reason, again, is that this Hamiltonian can be obtained from the non-interacting Hamiltonian by making a transformation - $\psi_{\rm RS}(\mathbf{r}_1 - \mathbf{r}_2) = (\mathbf{r}_1 - \mathbf{r}_2)\psi(\mathbf{r}_1 - \mathbf{r}_2)$, although, the transformation is not pure gauge.)

However, the composite fermion interacting with the RS gauge field as in Eq.(1) is *not* merely a transform of the non-interacting fermion. This is reflected in the two body problem explicitly since the wave-functions and energy eigenvalues are now different. Even after a redefinition of the angular momentum, the radial part of the wave-function is still different. Most interestingly, the massive degeneracy with respect to angular momentum has disappeared. From the energy expression in Eq.(8), we see that the energy is minimised when $(L - \alpha/\pi)$, or equivalently L, is maximised. In the $L \to \infty$ limit, the energy eigenvalue attains its minimum of $eB/2m$.

For the two CF system that we have solved explicitly, the maximum value of the angular momentum is fixed by the size of the system . $L_{\max} = mvR$ where R is the size of the system and plugging in the limiting value of v, we obtain $L_{\max} = R/l$ - *i.e.*, the size of the system measured in units of magnetic length. So explicitly for two composite fermions, we obtain the following result for the ground state -

$$R(r) = r^s e^{-r^2/4l^2} F(\frac{1}{2}[s' - (L_{\max} - \alpha/\pi) + 1 - mEl^2], s' + 1, \frac{r^2}{2l^2}) \quad (13)$$

$$E_0 = \frac{eB}{2m}(s' - (L_{\max} - \alpha/\pi) + 1) \quad (14)$$

where $s' = \sqrt{(L_{\max} - \alpha/\pi)^2 + (\alpha/\pi)^2}$ and $L_{\max} = R/l$.

Generalising this result to a system of many composite fermions, we see that the energy will be minimised if the relative angular momentum between any pair of composite fermions takes the maximum value that it can, given the size or equivalently, the density of particles in the sample. For instance, for FQHE at the fraction ν, the ratio $R/l = 1/\nu$, where we interpret R as the average distance between the composite fermions. This clearly shows that $L^{\rm rel}_{\max} = 1/\nu$ and energetically, this will be the favoured relative angular

momentum. A further assumption of analyticity, (lowest Landau level condition), leads directly to the Laughlin wave-functions. The same argument of maximising relative angular momentum also justifies the addition of even number of vortices in the Jain wave-functions. (Note however, that the solution for two composite fermions is *not* analytic because of the square root factor. Hence, naive generalisation of the wave-function for two composite fermions does not lead to the correct many-body wave-functions.)

The question of why Coulomb interactions disguise themselves as vortices attached to the fermions still remains open. However, we have now proved that if vortices are attached to fermions, then they like to maximise their relative angular momenta and stay as far apart as possible, thus minimising their Coulomb energy. In other words, we have shown that formation of composite fermions mimics the effect of a Coulomb potential in that it makes it energetically favourable for the fermions to stay apart.

We have taken the RS gauge potential as being the most relevant piece of the RS field theory and studied composite fermions in terms of the normal Hamiltonian that one would write down in the presence of such a complex gauge potential. Our results, thus, confirm the idea that the RS formulation of composite fermions is a much better starting point for FQHE than the CS formulation, since the RS gauge field incorporates Coulomb repulsions. Incidentally, perhaps, that is also why the RS formulation is able to obtain the Laughlin wave-function at the mean field level. Perturbation theory about the RS mean field FQHE state is more likely to be stable, since incorporation of repulsions implies that the FQHE gap is already formed at the mean field level. The drawback is that perturbation theory is more difficult in the RS theory, due to the non-hermiticity of the mean field Hamiltonian. Recently, however, a consistent perturbation theory has been formulated[12].

In conclusion, let us reiterate the main results of this letter. We have shown that for two composite fermions in an external magnetic field, the maximum possible value of the relative angular momentum is energetically favoured. This result required the input of the amplitude of the vortex attached to the fermion. The pure phase part of the vortex, which is what is captured in the CS formulation of FQHE is not sufficient to break the angular momentum degeneracy. With this result, it is easy to see why composite fermions work so well at minimising the Coulomb energy. Although, numerically, it is well-known that composite fermions minimise Coulomb energy, this is the first analytic proof of the result.

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